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THE KS METHOD IN LIGHT OF GENERALIZED EULER PARAMETERS.(U)

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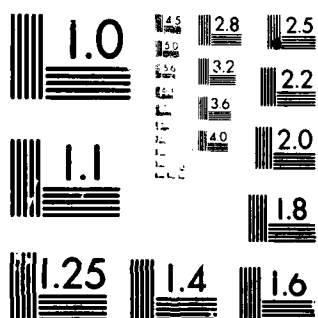
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or

$$\begin{matrix} r' \\ \cong \\ \cong \\ \cong \end{matrix} = R(q) \begin{matrix} r \\ \cong \\ \cong \\ \cong \end{matrix} \quad (25b)$$

A negative rotation of the vector (or positive rotation of axes) produces the transpose of the matrix. This can be seen directly since pairs of q's in the off diagonal terms which do not contain  $q_0$  are even in  $\theta$  while those containing  $q_0$  are odd.

A triad of orthogonal vectors can be introduced by

$$x = \begin{pmatrix} q_1 - q_2 - q_3 + q_0 \\ 2(q_1 q_2 - q_3 q_0) \end{pmatrix}, \quad y = \begin{pmatrix} 2(q_1 q_2 + q_3 q_0) \\ -q_1^2 + q_2^2 - q_3^2 + q_0^2 \end{pmatrix}, \quad z = \begin{pmatrix} 2(q_1 q_3 - q_2 q_0) \\ 2(q_1 q_0 + q_2 q_3) \end{pmatrix} \quad (26)$$

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## FOREWORD

The work represented by this technical report was performed in the Fire Control Presetting Analysis Branch of the FBM Geoballistics Division, Strategic Systems Department under Task Assignment No. 37430.

This report has been reviewed by J. R. Fallin and D. L. Owen of the FBM Geoballistics Division.

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Strategic Systems Department

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## INTRODUCTION

Stiefel and Scheifele<sup>1</sup> have given a general discussion of the Kustaanheimo-Stiefel<sup>2</sup> transformation method (KS) in the two-body problem. Many papers have appeared in which specific problems or applications have been addressed; but the feeling persists that in going from a three-space to a four-space representation, fundamental understanding is impaired. Thus, there seems to be a reluctance to apply this method to trajectory determination, guidance algorithms, or rigid-body kinematics in spite of favorable claims by some authors with regard to its efficiency and accuracy.<sup>3,4,5</sup>

It is the objective of this paper to show that the KS method can be understood in terms of a generalized Euler parameterization of rotations and dilations. The second section, EQUATION OF MOTION, sets up the equation of motion for the restricted two-body problem while the third section, KS TRANSFORMATION MATRIX, briefly reviews the KS method and applies it to the two-body equation of motion. The fourth section, ROTATION ABOUT AN ARBITRARY AXIS, sets up the Euler parameter description of rotations and then generalizes it to include dilations. Finally, the kinematical equations of this representation are compared to the KS dynamical equations of motion to show that they have equivalent interpretations.

## EQUATION OF MOTION

The equation of motion per unit mass in the restricted case is

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3} \mathbf{r} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \quad (1)$$

where  $\mu = GM$  and  $\mathbf{F}$  is the perturbing acceleration. If  $E$  is the eccentric anomaly, then consider a scaled eccentric anomaly defined by

$$s = \frac{E}{v_m} \quad (2)$$

where  $V_m$  is the geometric mean of the Keplerian velocities at apocenter and pericenter. Then a change of variables can be made with

$$dt = r ds \quad (3)$$

so that the equations of motion become

$$\frac{1}{r^2} \frac{d^2 r}{ds^2} - \frac{r}{r^3} \frac{d^2 r}{ds^2} + \frac{\mu}{r^3} r = F(r, \frac{d^2 r}{ds^2}, s) \quad (4)$$

where  $\frac{d}{ds}$  denotes the derivative with respect to  $s$ . In the case where  $F=0$  (the Kepler problem), the energy  $E_k$  is a constant given by

$$E_k = E_{\text{kinetic}} + E_{\text{potential}} = \frac{v^2}{2} - \frac{\mu}{r} = -2\alpha_k \quad (5)$$

where  $\alpha_k > 0$  for elliptical orbits, the only case of interest here.

#### KS TRANSFORMATION MATRIX

P. Kustaanheimo and E. Stiefel<sup>2</sup> proposed a regularization method by introducing a  $4 \times 4$  transformation matrix and four-component vector given by

$$L(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \quad (6)$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

such that  $\underset{\cong}{r} = \underset{\cong}{L(u)} \underset{\cong}{u}$ ; that is

$$\underset{\cong}{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \underset{\cong}{L(u)} \underset{\cong}{u} = \begin{pmatrix} u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ 2(u_1 u_2 - u_3 u_4) \\ 2(u_1 u_3 + u_2 u_4) \\ 0 \end{pmatrix} \quad (7)$$

and  $|\underset{\cong}{r}| = \underset{\cong}{u} \cdot \underset{\cong}{u} = u^2$ . Since  $\underset{\cong}{L(u)}$  is an orthogonal matrix, the inverse  $\underset{\cong}{L}^{-1}(u)$  is related to the transpose  $\underset{\cong}{L}^T(u)$  by

$$\underset{\cong}{L}^{-1}(u) = \frac{1}{u^2} \underset{\cong}{L}^T(u) \quad (8)$$

Thus

$$\underset{\cong}{\ddot{u}} = \frac{1}{2u^2} \underset{\cong}{L}^T(u) \underset{\cong}{F} \quad (9)$$

with  $\underset{\cong}{u}$  and  $\underset{\cong}{\ddot{u}}$  satisfying a bilinear relation, denoted  $\underset{\cong}{\ell}(u, \underset{\cong}{\ddot{u}}) = 0$ , such that

$$\underset{\cong}{\ell}(u, \underset{\cong}{\ddot{u}}) = u_4 \underset{\cong}{\ddot{u}}_1 - u_3 \underset{\cong}{\ddot{u}}_2 + u_2 \underset{\cong}{\ddot{u}}_3 - u_1 \underset{\cong}{\ddot{u}}_4 = 0 \quad (10)$$

and  $\underset{\cong}{\ell}(u, \underset{\cong}{\ddot{u}})$  is a constant of the motion in the perturbed two-body problem. The details of this method have been discussed by E. Stiefel and G. Scheifele.<sup>1</sup>

Introduction of the KS transformation, equation (7), into the equations of motion (4) leads to the u-space equation of motion

$$\underset{\cong}{\ddot{u}} + \left( \frac{\mu}{2r} - \frac{\ddot{u}^2}{r} \right) \underset{\cong}{u} = \frac{r}{2} \underset{\cong}{L}^T(u) \underset{\cong}{F} \quad (11)$$

which with the energy relation (5) becomes

$$\frac{u}{\approx} + \alpha_k \frac{u}{\approx} = \frac{r}{2} \frac{L^T(u)}{\approx} F \quad (12)$$

Recall that in the Keplerian problem the potential energy is

$$E_{\text{potential}} = V_{\text{kep}} = - \frac{\mu}{r}$$

so that the total Keplerian energy is

$$E_{\text{kep}} = E_{\text{kin.}} + V_{\text{kep}} = \frac{v^2}{2} - \frac{\mu}{r} = - 2\alpha_k$$

The total energy  $E_t$  can be written as

$$E_t = E_{\text{kin.}} + E_{\text{pot.}} = \frac{v^2}{2} + V_{\text{kep}} + V_{\text{pG}} + V_p = \frac{v^2}{2} + V_t$$

where

$$V_{\text{kep}} = - \mu/r$$

$$V_{\text{pG}} = \text{Potential due to perturbing (non-spherical) gravity}$$

$$V_p = \text{Potential due to non-gravitational perturbations}$$

$$V_t = \text{Total Potential} = V_{\text{kep}} + V_{\text{pG}} + V_p$$

Then  $E_t$  can be written as

$$E_t = -2\alpha_k + V_{\text{pG}} + V_p$$

With the definition  $2\alpha_0 \equiv -E_t$  the above becomes

$$\alpha_0 = \alpha_k - \frac{1}{2} V_{\text{pG}} - \frac{1}{2} V_p \quad (13)$$

Now the total perturbing acceleration  $F$  can be split into that derivable from a potential  $V_{tp}$  plus that which is not (P), so

$$\begin{aligned} F &= - \partial_r V_{tp} + P \\ &= - \partial_r V_{pG} - \partial_r V_p + P \end{aligned}$$

The equations of motion (12) then become

$$\ddot{u} + \alpha_0 u = - \frac{1}{2} V_{pG} u - \frac{1}{2} V_p u + \frac{r}{2} L^T(u) F$$

or

$$\ddot{u} + \alpha_0 u = - \frac{1}{2} V_{pG} u - \frac{1}{2} V_p u + Q$$

where

$$Q = \frac{r}{2} L^T(u) F = - \frac{r}{2} L^T(u) \partial_r V_{tp} + \frac{r}{2} L^T(u) P$$

It can be seen that

$$L^T(u) \partial_r = \frac{1}{2} \partial_u V$$

by expanding both sides, so that the equations of motion become

$$\ddot{u} + \alpha_0 u = - \frac{1}{2} V_{pG} u - \frac{1}{2} V_p u + \frac{r}{2} \left( - \frac{1}{2} \partial_u V_{tp} + L^T(u) P \right) \quad (14)$$

If all perturbations are derivable from a potential  $V_{tp}$ , then  $P = 0$  and no terms with  $L^T(u)$  appear in the equations of motion giving

$\approx$

$$\ddot{u} + \alpha_0 u = -\frac{1}{2} V_{pG} u - \frac{1}{2} V_p u - \frac{r}{4} \partial_u V_{pG} - \frac{r}{4} \partial_u V_p \quad (15)$$

If all perturbations are due to the gravitational field, then  $V_p = 0$  and

$$\ddot{u} + \alpha_0 u = -\frac{1}{2} V_{pG} u - \frac{r}{4} \partial_u V_{pG} \quad (16)$$

Finally, some authors use  $u^* = \frac{du}{dE}$  so that with  $\omega_0^2 = \alpha_0$

$$\ddot{u} = 2\omega_0 u^*$$

and equation (16) becomes

$$u^{**} + \frac{1}{4} u = \frac{-1}{8\omega_0^2} (V_{pG} u + \frac{r}{2} \partial_u V_{pG}) \quad (17)$$

Using equation (12) for the equation of motion, a substitution  $\alpha_k = \omega_k^2$  and

$\frac{r}{2} L^T(u)F = D(u)$  gives the form

$$\ddot{u} + \omega_k^2 u = D(u) \quad (18)$$

which is the form for driven, coupled oscillators. If  $u^{(0)}$  is a solution for

$D(u) = 0$ , the homogeneous solution is

$$u^{(0)}(s) = u^{(0)}(0) \cos(\omega_k s) + \frac{1}{\omega_k} \dot{u}^{(0)}(0) \sin(\omega_k s) \quad (19)$$

while the general solution is the integral equation

$$\begin{aligned} \underline{u}(s) = \underline{u}^{(0)}(s) + \frac{1}{\omega_k} \sin(\omega_k s) \int_0^s \cos(\omega_k \sigma) \underline{D}(\underline{u}(\sigma)) d\sigma \\ - \frac{1}{\omega} \cos(\omega_k s) \int_0^s \sin(\omega_k \sigma) \underline{D}(\underline{u}(\sigma)) d\sigma \end{aligned} \quad (20)$$

A method of solution is to use the unperturbed solution  $\underline{u}^{(0)}(s)$  in the integrands, the first Born approximation, leading to a set of elementary integrals to evaluate. Andrus<sup>3</sup> has done this with  $\underline{D}$  for the gravitational perturbation acceleration including oblateness.

#### ROTATION ABOUT AN ARBITRARY AXIS

If a vector  $\underline{r}$  is rotated positively about an axis  $\hat{n}$  thru an angle  $\theta$ , then the resulting vector  $\underline{r}'$  is given by

$$\underline{r}' = \underline{r} \cos \theta + (1 - \cos \theta)(\hat{n} \cdot \underline{r}) \hat{n} - \sin \theta (\underline{r} \times \hat{n}) \quad (21)$$

This is equivalent to a negative rotation of the axis. The former rotation is referred to as an active rotation while the latter a passive rotation. If a coordinate system is introduced such that

$$\hat{n} = (\cos \alpha, \cos \beta, \cos \gamma)$$

$$\underline{r} = (x_1, x_2, x_3)$$

then equation (21) becomes

$$\underline{r'} = \left[ \underline{I} - 2 \sin^2 \frac{\theta}{2} \underline{A}(\alpha, \beta, \gamma) + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \underline{B}(\alpha, \beta, \gamma) \right] \underline{r} \quad (22)$$

where

$$\underline{A}(\alpha, \beta, \gamma) = \begin{pmatrix} \sin^2 \alpha & -\cos \alpha \cos \beta & -\cos \alpha \cos \gamma \\ -\cos \alpha \cos \beta & \sin^2 \beta & -\cos \beta \cos \gamma \\ -\cos \alpha \cos \gamma & -\cos \beta \cos \gamma & \sin^2 \gamma \end{pmatrix} \quad (23)$$

$$\underline{B}(\alpha, \beta, \gamma) = \begin{pmatrix} 0 & -\cos \gamma & \cos \beta \\ \cos \gamma & 0 & -\cos \alpha \\ -\cos \beta & \cos \alpha & 0 \end{pmatrix}$$

and  $\underline{I}$  is the unit matrix. Equation (22) is the Whittaker<sup>6</sup> result. Introducing the Euler parameters

$$\begin{aligned} q_0 &= \cos \frac{\theta}{2} \\ q_1 &= \cos \alpha \sin \frac{\theta}{2} \\ q_2 &= \cos \beta \sin \frac{\theta}{2} \\ q_3 &= \cos \gamma \sin \frac{\theta}{2} \end{aligned} \quad (24)$$

with  $\sum q_\lambda^2 = 1$ , the above becomes

$$\underline{r'} = \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1 q_2 - q_3 q_0) & 2(q_1 q_3 + q_2 q_0) \\ 2(q_1 q_2 + q_3 q_0) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(q_2 q_3 - q_1 q_0) \\ 2(q_1 q_3 - q_2 q_0) & 2(q_2 q_3 + q_1 q_0) & -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{pmatrix} \underline{r} \quad (25a)$$

or

$$\begin{matrix} \underline{r'} \\ \cong \\ \cong \\ \cong \end{matrix} = \underline{R(q)} \begin{matrix} \underline{r} \\ \cong \\ \cong \\ \cong \end{matrix} \quad (25b)$$

A negative rotation of the vector (or positive rotation of axes) produces the transpose of the matrix. This can be seen directly since pairs of  $q$ 's in the off diagonal terms which do not contain  $q_0$  are even in  $\theta$  while those containing  $q_0$  are odd.

A triad of orthogonal vectors can be introduced by

$$\begin{matrix} \underline{x} \\ \cong \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} q_1 - q_2 - q_3 + q_0 \\ 2(q_1 q_2 - q_3 q_0) \\ 2(q_1 q_3 + q_2 q_0) \end{pmatrix}, \begin{matrix} \underline{y} \\ \cong \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} 2(q_1 q_2 + q_3 q_0) \\ -q_1^2 + q_2^2 - q_3^2 + q_0^2 \\ 2(q_2 q_3 + q_1 q_0) \end{pmatrix}, \begin{matrix} \underline{z} \\ \cong \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} 2(q_1 q_3 - q_2 q_0) \\ 2(q_2 q_3 + q_1 q_0) \\ -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{pmatrix} \quad (26)$$

Thus equation (25) represents a projection of  $\underline{r}$  onto the rotated axes  $\underline{x}, \underline{y}, \underline{z}$  and the rotation is generated by

$$\begin{matrix} \underline{x} \\ \cong \\ \cong \\ \cong \end{matrix} = \underline{\mathcal{L}(q)} \begin{matrix} \underline{q} \\ \cong \\ \cong \\ \cong \end{matrix}, \begin{matrix} \underline{y} \\ \cong \\ \cong \\ \cong \end{matrix} = \underline{\mathcal{L}(q)} \begin{matrix} \underline{p} \\ \cong \\ \cong \\ \cong \end{matrix}, \begin{matrix} \underline{z} \\ \cong \\ \cong \\ \cong \end{matrix} = \underline{\mathcal{L}(q)} \begin{matrix} \underline{s} \\ \cong \\ \cong \\ \cong \end{matrix} \quad (27)$$

where

$$\begin{matrix} \underline{q_a} \\ \cong \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_0 \end{pmatrix}, \begin{matrix} \underline{q_b} \\ \cong \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} q_2 \\ -q_1 \\ -q_0 \\ q_3 \end{pmatrix}, \begin{matrix} \underline{q_c} \\ \cong \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} q_3 \\ q_0 \\ -q_1 \\ -q_2 \end{pmatrix} \quad (28)$$

and

$$\begin{matrix} \underline{\mathcal{L}} \\ \cong \\ \cong \\ \cong \end{matrix} (q) = \begin{pmatrix} q_1 & -q_2 & -q_3 & q_0 \\ q_2 & q_1 & -q_0 & -q_3 \\ q_3 & q_0 & q_1 & q_2 \\ -q_0 & q_3 & -q_2 & q_1 \end{pmatrix} \quad (29)$$

Note the identity between the  $\underline{x}$  vector and the position vector  $\underline{r} = L(\underline{u})\underline{u}$ ,  
equation (7), of the KS transformation if  $q_i = u_i$  and  $q_0 = u_4$ ,  $i = 1, 2, 3$ . One  
could introduce two other vectors orthogonal to  $\underline{r}$  in the KS formalism as

$$\underline{r}_T = \begin{pmatrix} 2(u_1 u_2 + u_3 u_4) \\ u_4^2 - u_1^2 + u_2^2 - u_3^2 \\ 2(u_2 u_3 - u_1 u_4) \end{pmatrix}, \quad \underline{r}_N = \begin{pmatrix} 2(u_1 u_3 - u_2 u_4) \\ 2(u_2 u_3 + u_1 u_4) \\ u_4^2 - u_1^2 - u_2^2 + u_3^2 \end{pmatrix} \quad (30)$$

and generated by

$$\underline{r}_T = L(\underline{u}) \begin{pmatrix} u_2 \\ -u_1 \\ -u_4 \\ u_3 \end{pmatrix}, \quad \underline{r}_N = L(\underline{u}) \begin{pmatrix} u_3 \\ u_4 \\ -u_1 \\ -u_2 \end{pmatrix} \quad (31)$$

and interpret the KS formalism as a parameterization of a rotated set of axes onto which an initial vector is projected to obtain the resulting vector.

For completeness, a fourth vector

$$\underline{q}_d = \begin{pmatrix} q_0 \\ q_3 \\ -q_2 \\ q_1 \end{pmatrix} \quad (32)$$

can be introduced which is orthogonal to  $\underline{q}_a$ ,  $\underline{q}_b$  and  $\underline{q}_c$ . These orthogonal vectors  
now span  $R^4$ . In particular, it is not hard to show that  $\underline{q}_a$  can be expanded with

components along  $\underline{q}_b$ ,  $\underline{q}_c$  and  $\underline{q}_d$  which results in

$$\dot{\underline{q}}_a = \frac{1}{2} \underline{B}(\underline{\omega}) \underline{q}_a \quad (33)$$

where

$$\underline{B}(\underline{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & \omega_1 \\ \omega_3 & 0 & -\omega_1 & \omega_2 \\ -\omega_2 & \omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{pmatrix} \quad (34)$$

and  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$  is the angular velocity vector and  $\omega_i = \dot{\theta}_i$ . Note that equation (33) can also be written as

$$\dot{\underline{q}}_a = \frac{1}{2} \begin{pmatrix} q_0 & q_3 & -q_2 & q_1 \\ -q_3 & q_0 & q_1 & q_2 \\ q_2 & -q_1 & q_0 & q_3 \\ -q_1 & -q_2 & -q_3 & q_0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{pmatrix} \quad (35)$$

From equation (33) it is seen that with  $\underline{q} \leftrightarrow \underline{q}_a$

$$\ddot{\underline{q}} = \frac{1}{2} \left( \dot{\underline{B}} \underline{q} + \underline{B} \dot{\underline{q}} \right) = \frac{1}{2} \left( \dot{\underline{B}} \underline{q} + \frac{1}{2} \underline{B}^2 \underline{q} \right)$$

But with the fact that  $\underline{B}^2 = -\omega^2 \underline{I}$ , there results

$$\ddot{\underline{q}} + \frac{\omega^2}{4} \underline{q} = \frac{1}{2} \dot{\underline{B}} \underline{q} \quad (36)$$

Recall the change of variable introduced in equation (3) giving for a particle a distance  $r$  from the origin

$$\dot{q} = \frac{1}{r} \dot{Q}$$

If, for the time being,  $r$  is considered constant for pure rotations of the coordinate system or circular motion of the particle, equation (36) becomes

$$\ddot{q} + \frac{r^2 \omega^2}{4} q = \frac{r}{2} \ddot{Q} \quad (37)$$

But for pure rotations, the total energy is composed of the rotational energy  $\frac{1}{2} r^2 \omega^2$  and the potential energy  $-\mu/r$ , so there results

$$E_{\text{kep}} = -2\alpha_k = \frac{1}{2} r^2 \omega^2 - \mu/r = \frac{1}{2} r^2 \omega^2 - r^2 \omega^2 = -\frac{r^2}{2} \omega^2$$

or

$$\alpha_k = \frac{r^2 \omega^2}{4}$$

and equation (37) becomes

$$\ddot{q} + \alpha_k \ddot{Q} = \frac{r}{2} \ddot{Q} \quad (38)$$

which has the same form as equation (12). The case where  $\dot{r} \neq 0$  will be addressed later. However, at this point, it is suspected that the KS method is similar to the Euler parameterization of rotations, and that  $\ddot{Q}$  is related to the perturbations.

Another representation of equation (25) is obtained by introducing the quaternion notation such that

$$\begin{aligned} Q &\equiv q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = q_0 + \underline{q} \\ r &= 0 + x_1 e_1 + x_2 e_2 + x_3 e_3 = 0 + \underline{x} \\ Q^{-1} &= q_0 - \underline{q} \end{aligned} \quad (39)$$

Using quaternion multiplication, equation (25) becomes

$$\underline{r'} = Q \underline{r} Q^{-1} \quad (40)$$

Note that  $\hat{n}$  is an eigenvector of the rotation matrix  $R(\underline{q})$ ; that is

$$R(\underline{q}) \hat{n} = \hat{n} \quad (41)$$

or in quaternion representation

$$Q \hat{n} Q^{-1} = \hat{n} \quad (42)$$

Recall that

$$\sum q_\lambda^2 = 1 \quad \lambda = 0, 1, 2, 3$$

so that the transformation represents pure rotations. This can be generalized<sup>7</sup> by introducing

$$u_\lambda = \delta q_\lambda \quad (43)$$

with  $u^2 = \sum u_\lambda^2 = \delta^2$ . Then introducing the quaternion

$$U = u_0 + \underline{u} = \delta(q_0 + \underline{q}) = \delta Q \quad (44)$$

$$U^{-1} = \frac{1}{U^2} (u_0 - \underline{u})$$

there results

$$\underline{r}' = U \underline{r} U^{-1} \quad (45)$$

and the transformations now contain rotations and dilations. Using the matrix representation, equation (22) can be written as

$$\underline{r}' = R(\underline{u}) \underline{r} \quad (46)$$

where  $R(\underline{u})$  is the matrix in equation (25) with  $q_\lambda = u_\lambda$  and without the restriction that  $\sum u_\lambda^2 = 1$ . Also, if  $u_4$  is associated with  $q_0$  in the  $\underline{x}$  vector in equation (26), there results the  $\underline{r}$  vector, equation (7), of the KS method. Thus it may be considered that the KS method is, in essence, an Euler parameterization of transformation, generalized in the sense of Velte.<sup>7</sup> The transformations are generated in the KS method by

$$\underline{r} = L(\underline{u}) \underline{u},$$

in this paper by

$$\underline{x} = \mathcal{L}(\underline{u}) \underline{u}$$

or as in Vitins<sup>8</sup> by

$$\begin{matrix} \mathbf{x} \\ \cong \\ \cong \\ \cong \end{matrix} = \Lambda(\mathbf{u}) \begin{matrix} \mathbf{u} \\ \cong \\ \cong \end{matrix} \quad (47)$$

where

$$\begin{matrix} \Lambda(\mathbf{u}) \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & u_4 & u_3 \\ u_3 & -u_4 & u_1 & -u_2 \\ u_4 & u_3 & -u_2 & -u_1 \end{pmatrix} \quad (48)$$

$$\begin{matrix} \mathbf{u} \\ \cong \\ \cong \end{matrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

The equation of motion for  $\mathbf{q}$  (pure rotation) is given by equation (37). With  $\mathbf{u}_\lambda = \delta \mathbf{q}_\lambda$ , the equation of motion for  $\mathbf{u}$  (which now includes rotations and dilations) becomes

$$\begin{matrix} \frac{\partial^2}{\partial t^2} \mathbf{u} \\ \cong \\ \cong \end{matrix} + \frac{r^2 \omega^2}{4} \begin{matrix} \mathbf{u} \\ \cong \\ \cong \end{matrix} = \left[ \frac{\partial^2}{\partial t^2} \mathbf{I} + 2 \frac{\partial}{\partial t} \mathbf{B}(\omega) + \frac{r}{2} \mathbf{B}(\omega) \right] \begin{matrix} \mathbf{u} \\ \cong \\ \cong \end{matrix} \quad (49)$$

For pure rotations  $\delta = \text{constant}$  and equation (49) reduces to

$$\begin{matrix} \frac{\partial^2}{\partial t^2} \mathbf{u} \\ \cong \\ \cong \end{matrix} + \frac{r^2 \omega^2}{4} \begin{matrix} \mathbf{u} \\ \cong \\ \cong \end{matrix} = \frac{r}{2} \mathbf{B}(\omega) \begin{matrix} \mathbf{u} \\ \cong \\ \cong \end{matrix}$$

or

$$\begin{matrix} \frac{\partial^2}{\partial t^2} \mathbf{q} \\ \cong \\ \cong \end{matrix} + \frac{r^2 \omega^2}{4} \begin{matrix} \mathbf{q} \\ \cong \\ \cong \end{matrix} = \frac{r}{2} \mathbf{B}(\omega) \begin{matrix} \mathbf{q} \\ \cong \\ \cong \end{matrix}$$

which is equation (37). Using the relation

$$\frac{r^2 \omega^2}{4} = \alpha_k - \varepsilon$$

where

$$\varepsilon = \frac{c^2}{4} \left( \frac{1}{b^2} - \frac{1}{r^2} \right)$$

$c$  = area constant

$b$  = semiminor axis

equation (49) becomes

$$\underset{\cong}{\overset{00}{u}} + \underset{\cong}{\alpha_k} \underset{\cong}{u} = \left[ \underset{\cong}{\overset{00}{\delta/\delta}} \underset{\cong}{I} + 2 \underset{\cong}{\overset{0}{\delta\delta B}(\omega)} + \underset{\cong}{\frac{r}{2}} \underset{\cong}{\overset{0}{B}(\omega)} + \Lambda \right] \underset{\cong}{u} \quad (50)$$

Comparing equation (50) with equation (12) it is seen that the kinematical equation due to the Euler parameterization of rotations and dilations has the same form as the dynamical equation of the KS method.

### CONCLUSION

A simple interpretation of the KS method can be made; namely, the method is equivalent to treating the dynamics in a rotating and dilating coordinate system characterized by generalized Euler parameters.

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